Tut 8

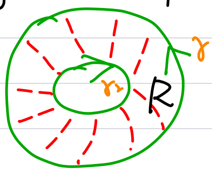
Green's thm: $\displaystyle\int_{\partial R} P\,dx + Q\,dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$

provided: ∂R is the disjoint union of simple closed curves.
(and R is contained in some open subset of $\mathbb{R}^2$ on which P, Q are smooth)

Special case: $\displaystyle\int_C x\,dy = -\int_C y\,dx = \int_R 1\,dx\,dy = \text{Area}(R)$

Rmk: Positively oriented means while you are traveling along the curve, R is on your left

Q1: Find, using Green's thm, the area of the region R bounded by the $x^2+y^2=4$ and $x^2+y^2=1$



Ans: Formula: $\displaystyle\text{Area}(R) = -\int_{\partial R} y\,dx$ (or $\int x\,dy$)

Step 1: Write done γ_1, γ_2

$\gamma_1(t) = (2\cos t, 2\sin t) \qquad 0 \le t \le 2\pi$

$\gamma_2(t) = (\cos(-t), \sin(-t)) \qquad 0 \le t \le 2\pi$

Step 2: $\displaystyle\text{Area} = \int_0^{2\pi} -2\sin t\, d\,2\cos t + \int_0^{2\pi} -\sin(-t)\, d\cos(-t)$

$\displaystyle= \int_0^{2\pi} 4\sin^2 t\,dt + \int_0^{2\pi} -\sin^2 t\,dt$

$\displaystyle= \int_0^{2\pi} 3\sin^2 t\,dt$

$\displaystyle= \frac{3}{2}\int_0^{2\pi} 1-\cos 2t\,dt = 3\pi$

Polar coordinates:

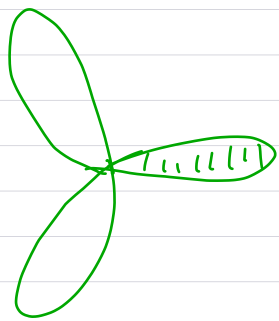
$$\text{Area}(R) = \int_{\partial R} x\,dy = \int_{\partial R} r\cos\theta\, d r\sin\theta = \int_{\partial R} r^2\cos^2\theta\, d\theta + r\cos\theta\sin\theta\, dr$$

$$\text{Area}(R) = -\int_{\partial R} y\,dx = -\int_{\partial R} r\sin\theta\, d r\cos\theta = \int_{\partial R} r^2\sin^2\theta\, d\theta - \int_{\partial R} r\sin\theta\cos\theta\, dr$$

$$2\text{Area}(R) = \int x\,dy - y\,dx = \int_{\partial R} r^2\, d\theta$$

$$\Longrightarrow \text{Area}(R) = \frac{1}{2}\int_{\partial R} r^2\, d\theta$$

Q2: Find the area of one leaf of the region bounded by $r = \cos 3\theta$.



Ans: $\text{Area} = \frac{1}{2}\int_{\partial R} r^2\, d\theta$

Step 1: $(r(t), \theta(t)) = (\cos 3t, t) \quad -\frac{\pi}{6} \leq t \leq \frac{\pi}{6}$

Step 2: $\text{Area}(R) = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2 3t\, dt$

$$= \frac{1}{4}\int_{-\pi/6}^{\pi/6} 1 + \cos 6t\, dt$$

$$= \frac{\pi}{12}$$

For $u: \mathbb{R}^2 \to \mathbb{R}$, C simple closed curve (+ve oriented)

$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

$\frac{\partial u}{\partial n} := \nabla u \cdot \vec{n}$ along C

Then $\int_C \frac{\partial u}{\partial n} ds = \int_C \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \cdot \vec{n} \, ds = \int_C -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy = \int_R \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} dx dy$

$= \int_R \Delta u \, dA$

Q3: Suppose $\Delta u = 0$. Let $C(r)$ denote the circle centred at the origin with radius r. (anti-clockwise)

Let $I(r) = \frac{1}{r}\int_{C(r)} u \, ds$, show that I is a constant.

Ans: Let $\gamma(t) = (r\cos t, r\sin t)$ $0 \le t \le 2\pi$

Then $I(r) = \frac{1}{r}\int_0^{2\pi} u(\gamma(t)) \cdot r \, dt = \int_0^{2\pi} u(r(t)) dt = \int_0^{2\pi} u(r\cos t, r\sin t) \, dt$

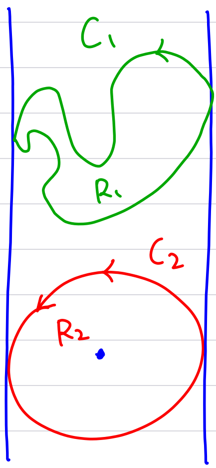
$\Rightarrow I'(r) = \int_0^{2\pi} u_x(\gamma(t)) \cos t + u_y(\gamma(t)) \sin t \, dt$

$= \int_0^{2\pi} \nabla u(\gamma(t)) \cdot \vec{n} \, dt$

$= \frac{1}{r}\int_0^{2\pi} \nabla u(r(t)) \cdot \vec{n} \, r dt$

$= \frac{1}{r}\int_{C(r)} \nabla u \cdot \vec{n} \cdot ds$

$= \frac{1}{r}\int_{\textcircled{r}} \Delta^2 u \, dA = 0$



$\gamma_1(t) = (x(t), y(t)) \quad 0 \le t \le 1$

$\gamma_2(t) = (\bar{x}(t), \bar{y}(t)) \quad 0 \le t \le 1$

Are parametrization of C_1 and C_2.

Q4: Suppose length$(C_1) = \ell$, Area$(R_1) = A$

C_2 is a circle of radius r, centred at the origin

Show that $A \le \frac{\ell^2}{4\pi}$

Ans: $A = \text{Area}(R_1) = \int_{C_1} x\,dy = \int_0^1 x(t)\,y'(t)\,dt$

$$\pi r^2 = \text{Area}(R_2) = -\int y\,dx = -\int_0^1 \bar{y}(t)\,\bar{x}'(t)\,dt$$

$$\Rightarrow \quad A + \pi r^2 = \int_0^1 x(t)\,y'(t) - \bar{y}(t)\,\bar{x}'(t)\,dt$$

Let $\vec{u}(t) = (\bar{x}(t), \bar{y}(t))$, $\vec{v}(t) = (y'(t), -x'(t))$,

Then
$$\begin{aligned} A + \pi r^2 &= \int_0^1 \vec{u}(t) \cdot \vec{v}(t)\,dt \\ &\le \int_0^1 |\vec{u}(t)|\,|\vec{v}(t)|\,dt \\ &= \int_0^1 r\,|\vec{v}(t)|\,dt \\ &= r\int_0^1 \sqrt{x'(t)^2 + y'(t)^2}\,dt \\ &= r\,\text{length}(C_1) = r\ell \end{aligned}$$

$$\Rightarrow \quad A + \pi r^2 \le \ell$$

Finally, AM-GM inequality $\Rightarrow \sqrt{A \cdot \pi r^2} \le \frac{A + \pi r^2}{2} \le \frac{1}{2} r\ell$

$$\Rightarrow \quad A \cdot \pi r^2 \le \frac{1}{4} r^2 \ell^2$$

$$\Rightarrow \quad A \le \frac{\ell^2}{4\pi}$$

More identities:

Q5: $\int_C \phi \nabla\phi \cdot \vec{n}\, ds = \iint_D (\phi \Delta\phi + \nabla\phi\cdot\nabla\phi)\, dA$ (i.e. $\int_C \phi \frac{\partial u}{\partial n} ds = \iint_D \phi\Delta\phi + \|\nabla\phi\|^2 dA$

proof: $\int_C \phi\nabla\phi\cdot\vec{n}\, ds = \int_C -\phi\phi_y dx + \phi\phi_x dy$

$$= \int_R \frac{\partial(\phi\phi_y)}{\partial y} + \frac{\partial(\phi\phi_x)}{dx}\, dA$$

$$= \int_R \phi_y\phi_y + \phi\phi_{yy} + \phi_x\phi_x + \phi\phi_{yy}\, dA$$

$$= \int_R \phi(\phi_{xx} + \phi_{yy}) + (\phi_x\phi_x + \phi_y\phi_y)\, dA$$

$$= \int_R \phi\Delta\phi + \nabla\phi\cdot\nabla\phi\, dA$$

Q 6: Let $f: S^1 \to \mathbb{R}$ be smooth. $(S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\})$

Show that if u, v are smooth function on $\mathbb{D} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$

Suppose $\Delta u = \Delta v = 0$

and $u = v = f$ on S

show that $u = v$

(Uniqueness of Dirichlet problem)

Answer: Let $w = u - v$, then $\Delta w = 0$ and $w = 0$ on S^1

Applying Q5 $\int_{S^1} \phi \frac{\partial \phi}{\partial n} ds = \int_{\mathbb{D}} \phi \Delta \phi + \|\nabla \phi\|^2 \, dA$ for $\phi = w$

We have $0 = \int_{S^1} w \frac{\partial w}{\partial n} ds = \int \|\nabla w\|^2 dA$

Since $\|\nabla w\|^2 \geq 0$, we must have $\nabla w \equiv 0$.

So w is constant.

But $w = 0$ on S^1, so $w \equiv 0$.

A similar result is: If $\Delta u = 0$ on $\mathbb{D}$ and $\frac{\partial u}{\partial n} = 0$ on S^1, then $u \equiv$ constant on D.

Q7: Prove $\int_C v \frac{\partial u}{\partial n} ds = \int_R \nabla v \cdot \nabla u + v \Delta u$

Ans: L.H.S. $\int_C -vu_y\, dx + vu_x\, dy = \int_R (vu_y)_y + (vu_x)_x\, dA$

$= \int_R v(u_{xx}+u_{yy}) + (u_x v_x + u_y v_y)$

$= \int_R v\Delta u + \nabla v \cdot \nabla u$

Q8: For a smooth function u on $\mathbb{D}$, we define the potential energy $E(u)$ of u by

$$E(u) = \frac{1}{2}\int_{\mathbb{D}} \|\nabla u\|^2 \, dA$$

Let $f: S^1 \to \mathbb{R}^1$ be a smooth function.

And suppose $u, v: \mathbb{D}^1 \to \mathbb{R}^1$ smooth and such that

$u = v = f$ on S^1

If $\Delta u = 0$ on $\mathbb{D}$, show that $E(v) \geq E(u)$.

Ans: Let $w = v - u$, then

$$E(v) = E(w+u) = \frac{1}{2}\int_{\mathbb{D}} |\nabla(w+u)|^2$$

$$= \frac{1}{2}\int_{\mathbb{D}} \|\nabla w\|^2 + \|\nabla u\|^2 + \nabla w \cdot \nabla u \quad dA$$

$$= E(u) + \frac{1}{2}\int_{\mathbb{D}} \|\nabla w\| \, dA + \frac{1}{2}\int_{R} \nabla w \cdot \nabla u \, dA$$

But $\int_{S^1} \nabla w \cdot \nabla u \, dA = \int_{S^1} w \frac{\partial u}{\partial n} \, ds - \int_{\mathbb{D}} v \Delta u \, dA$

$= 0 \qquad - \qquad 0$ (as $w = 0$ on S^1 and $\Delta u = 0$ on $\mathbb{D}$)

$= 0$